COMPARISON OF LOW-ORDER NONLINEAR AEROELASTIC MODELS FOR NUMERICAL CONTINUATION

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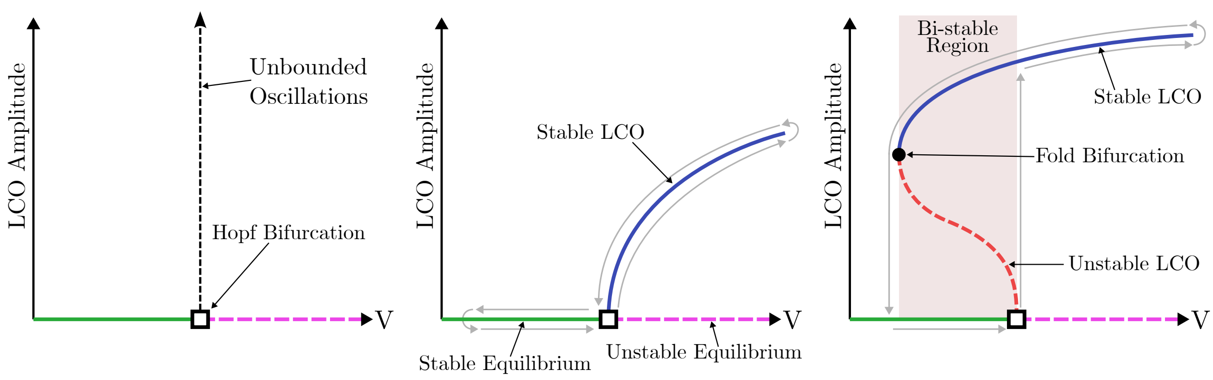
ABSTRACT

Highly flexible, high-aspect-ratio wings are a key enabler for next-generation, fuel-efficient aircraft, offering reduced induced drag and the potential to significantly reduce climate impact. However, their increased structural flexibility introduces strong nonlinear aeroelastic effects, including large static deflections, limit cycle oscillations (LCOs), and pose-dependent changes in stability [1–4]. Of particular importance are Hopf-type bifurcations, which arise when a complex conjugate pair of eigenvalues crosses the imaginary axis and mark the onset of an oscillatory - linear - instability. In linear theory, this corresponds to flutter with unbounded oscillatory growth at the point of instability (Fig. 1a), whereas in nonlinear systems the response is typically bounded, leading to finite-amplitude LCOs. These may emerge via either supercritical Hopf bifurcations, where oscillation amplitude grows smoothly from the Hopf with speed (Fig. 1b), or subcritical Hopf bifurcations, where large amplitude oscillations appear abruptly at the Hopf and persist over a hysteretic, bistable range of operating conditions (Fig. 1c). Whilst such behaviours are widely reported in the literature, there remains limited guidance on how to design cantilever wings that will deliberately exhibit (or, by proxy, reliably avoid) these nonlinear aeroelastic responses.

Numerical tools such as bifurcation analysis and numerical continuation [5–7] provide a promising framework for addressing this challenge, enabling systematic exploration of how nonlinear stability characteristics, including Hopf criticality, fold bifurcations, and nonlinear flutter onset speeds, evolve with changes in structural, aerodynamic, and operating parameters. In practice,



(a) A degenerative Hopf bifurcation (Linear Flutter).

(b) A supercritical Hopf bifurcation.

(c) A sub-critical Hopf bifurcation.

Figure 1: A diagrammatic representation of the three possible system responses at a Hopf bifurcation.

however, the effective use of these tools depends strongly on the choice of numerical model. Robust numerical continuation imposes specific requirements on the model formulation, including low state dimensionality, smooth parameter dependence, and a set of governing equations that can be described as a smooth autonomous dynamical system with no algebraic constraints (or only ones reducible to an autonomous evolution equation). Notably, these requirements closely overlap with those associated with gradient-based optimisation and parametric design studies, making continuation-ready aeroelastic models a natural candidate for both the analysis and design of nonlinear high-aspect-ratio wings.

This paper investigates several low-order structural and aerodynamic models to assess their suitability for numerical continuation of nonlinear aeroelastic systems. The focus is on model combinations that have: (i) the formulation of a smooth autonomous dynamical system; (ii) smooth parameter dependence suitable for both equilibrium and periodic continuation, as well as gradient-based exploration; and (iii) sufficient nonlinear structural and aerodynamic fidelity to capture the mechanisms governing aeroelastic behaviour. Three low-order nonlinear beam formulations are considered: the modal intrinsic beam (MIB) method [8], a nonlinear beam shape (NBS) formulation [9, 10], and a geometrically nonlinear Chebyshev–Ritz beam model [11]. These formulations represent different kinematic descriptions and treatments of geometric nonlinearity, spanning modal and non-modal representations, intrinsic and displacement-based kinematics, and varying orders of explicitly retained nonlinear terms.

Each beam formulation is coupled with a hierarchy of low-order unsteady aerodynamic models, including modified unsteady strip theory [12, 13], unsteady strip theory augmented with three-dimensional lifting-line corrections [14], and a simplified dynamic-stall formulation derived from the Beddoes–Leishman model [15, 16]. The resulting structural–aerodynamic combinations are compared not in terms of ultimate fidelity, but with respect to their suitability for low-order aeroelastic modelling. Specifically, their dimensionality, smoothness of parameter dependence, robustness for numerical continuation, and relative computational cost.

The aeroelastic models are assessed using three benchmark wing configurations: the classical Goland wing [17, 18], the PAZY wing [19, 20], and a highly flexible experimental wing developed at Bristol (SJD) [11]. These configurations were deliberately selected for their relatively simple geometries, well-documented structural properties, range of nonlinear behaviour, and availability of experimental aeroelastic data. Unlike many previous studies that focus on a single configuration and employ post hoc tuning of aerodynamic model parameters, this study uses a single set of aerodynamic parameters across all configurations to assess model robustness.

Numerical continuation (via equilibrium and periodic branch tracking) is employed to identify Hopf bifurcations, characterise LCO branches (including folds and the nonlinear flutter onset, NFOS), and assess model smoothness and robustness under parameter variation (e.g., airspeed, incidence, and structural properties). Time-marching simulations will also be used to qualitatively verify branch stability and transient response, as required.

Preliminary results from the modal intrinsic beam (MIB) formulation applied to the PAZY wing highlight the capability of low-order models to capture experimentally observed nonlinear behaviour. Static deflection predictions under tip loading show good agreement with other geometrically exact beam models (Fig. 2). Linear stability analysis reveals the presence of two distinct flutter mechanisms: a low–speed bending–torsion "hump" instability that re-stabilises at higher velocities, and a higher–speed bending–torsion flutter whose onset is relatively insensitive to angle of attack (Fig. 3). Bifurcation analysis at low incidence (Fig. 4) demonstrates

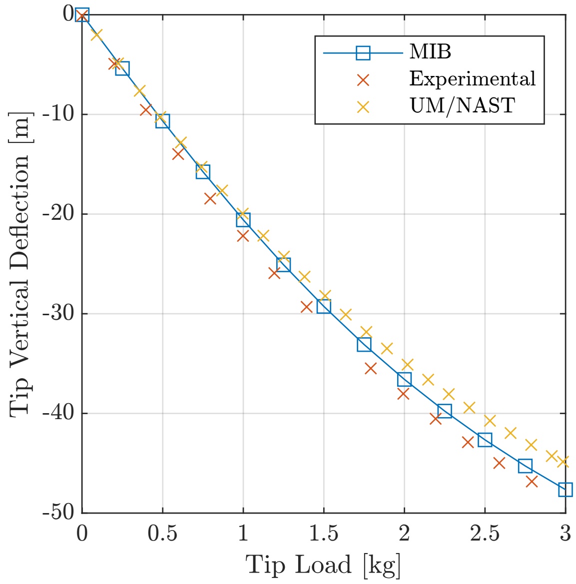


Figure 2: MIB - Variation in the tip deflection of the Pazy wing, under different tip masses.

that the first instability is subcritical, in qualitative agreement with experimental observations, while the second instability is supercritical, producing LCOs whose amplitude grows rapidly with increasing velocity.

Across the other benchmark cases considered, preliminary results indicate that the primary role of the structural beam formulation is to determine model order, numerical robustness, and computational efficiency, whereas the choice of aerodynamic model can significantly impact predicted instability mechanisms, Hopf criticality, and nonlinear stability boundaries.

The final paper will compare low-order structural and aerodynamic modelling approaches for nonlinear aeroelastic analysis using numerical continuation. The relative accuracy and computational efficiency of aero-structural model combinations will be assessed by comparing continuation-based predictions (equilibria, Hopf points, LCO branches, and folds) against three aeroelastic demonstrators: the classical Goland wing, the PAZY wing, and the SJD wing. These comparisons are intended to identify model combinations that are sufficiently robust and well posed to support the design and analysis of a future wind tunnel model that will exhibit prescribed nonlinear aeroelastic behaviour.

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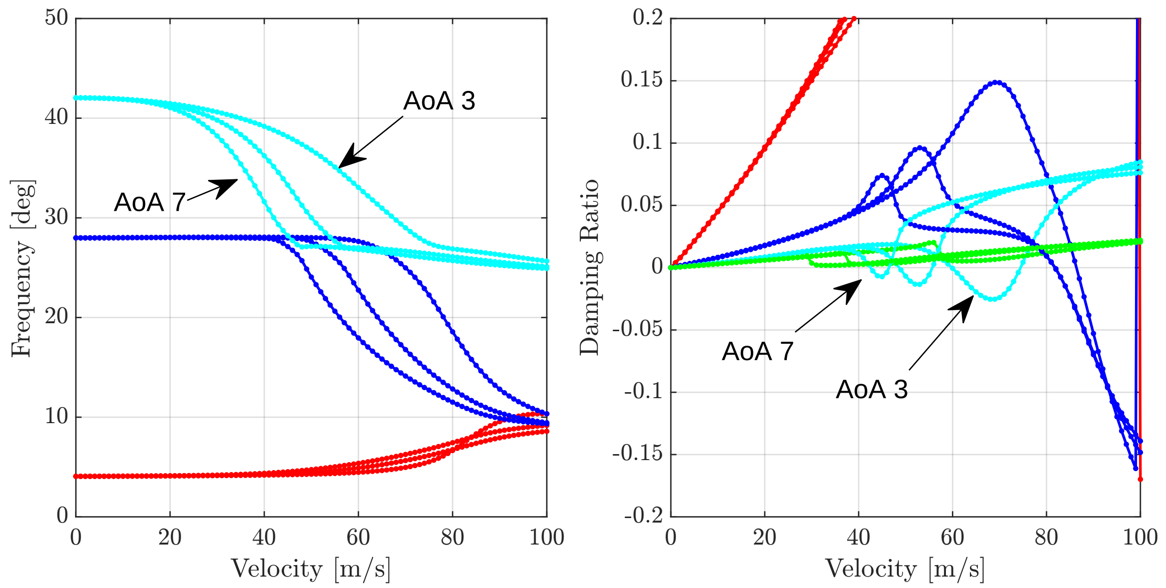

Figure 3: Variation in natural frequency and damping ratio of the Pazy wing at three AoAs.

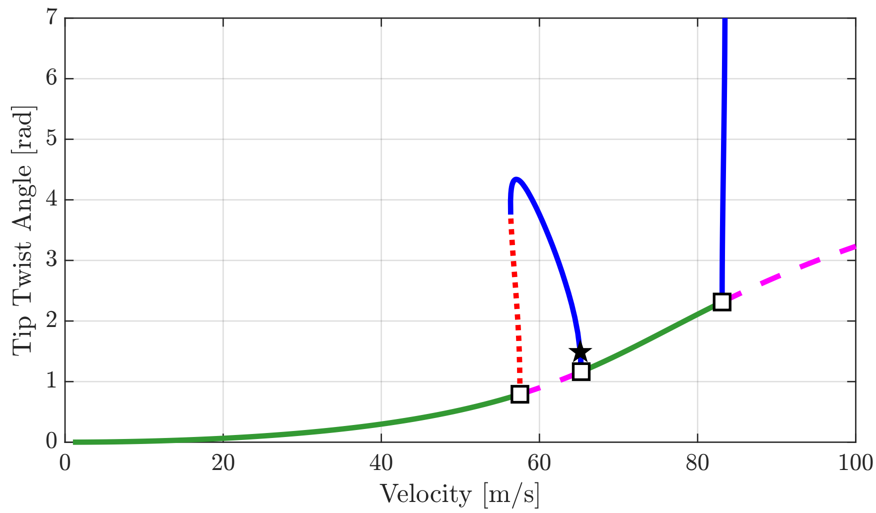

Figure 4: Variation in tip twist angle of fixed point and periodic solution branches for the Pazy wing at 1 deg AoA.

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